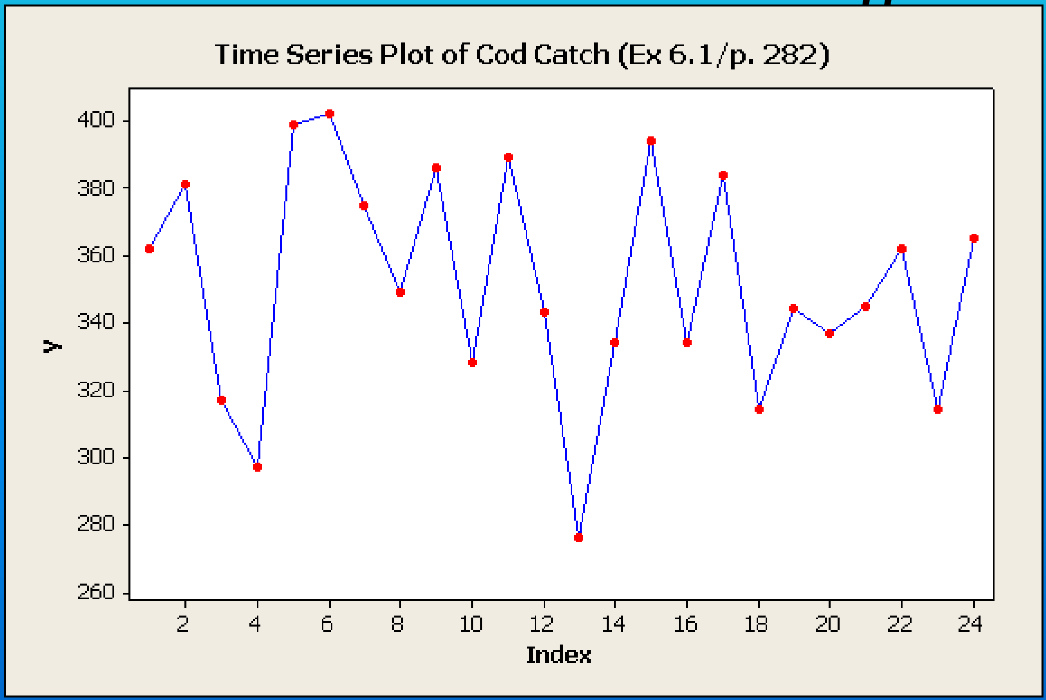
TIME SERIES

SIMPLE EXPONENTIAL SMOOTHING

•If the mean of y_t remains constant over time, each observation gets equal weight:

$$y_t = \beta_0 + e_t, \ \hat{\beta}_0 = \bar{y} = \frac{\sum_{t=1}^{n} y_t}{n}$$



•If the mean of y_t changes slowly over time, recent observations should have more weight than remote observations.

•Simple exponential smoothing can be used for forecasting such a series.

NOTATIONS USED:

y_t = value of the time series at time t

ℓ_t = level (mean) of the series at time t

SIMPLE EXPONENTIAL SMOOTHING:

(1) $\ell_0 = \dfrac{\sum_{t=1}^{n/2} y_t}{n}$ (mean of half of the data)

(2) $\ell_t = \alpha y_t + (1-\alpha)\ell_{t-1}, \quad 0 \le \alpha \le 1$ (Smoothing equation)

(3) Compute forecasts of historical data as follows:

$\hat{y}_1 = \hat{y}_{0+1}(0) = \ell_1$ = point forecast made at time 0 for value of next period, y_{0+1}

$\hat{y}_2 = \hat{y}_{1+1}(1) = \ell_2$ = point forecast made at time 1 for value of next period, y_{1+1}

$\hat{y}_3 = \hat{y}_{2+1}(1) = \ell_3$ = point forecast made at time 1 for value of next period, y_{2+1}

SIMPLE EXPONENTIAL SMOOTHING

(3) (continued)

For forecasts of historical data the general equation can be written as

$\hat{y}_{T+\tau} = \ell_T$ = point forecast made at time T

for value of next period, y_{T+1}

$T = 0, 1, 2, \ldots, n$ (historical data)

$\tau = 1, 2, 3, \ \ldots$

SIMPLE EXPONENTIAL SMOOTHING

(4) When $T = n$ (last historical data)

$$\hat{y}_{T+\tau} = \ell_T = \text{point forecast made at time } T$$

for value of next period, y_{T+1}

$\tau = 1, 2, 3, \ ...$

The prediction interval, however, will get wider as τ is increased (forecast made further in future)

A 95% prediction interval computed at time T for $y_{T+\tau}$ is $\ell_T \pm 1.96 \times s \times \sqrt{1+(\tau-1)\alpha^2}$ where

$$s = \sqrt{\frac{SSE}{n-1}} = \sqrt{\frac{\sum_{t=1}^{n}[y_t - \hat{y}_t(t-1)]^2}{n-1}} = \sqrt{\frac{\sum_{t=1}^{n}[y_t - \ell_{t-1}]^2}{n-1}}$$

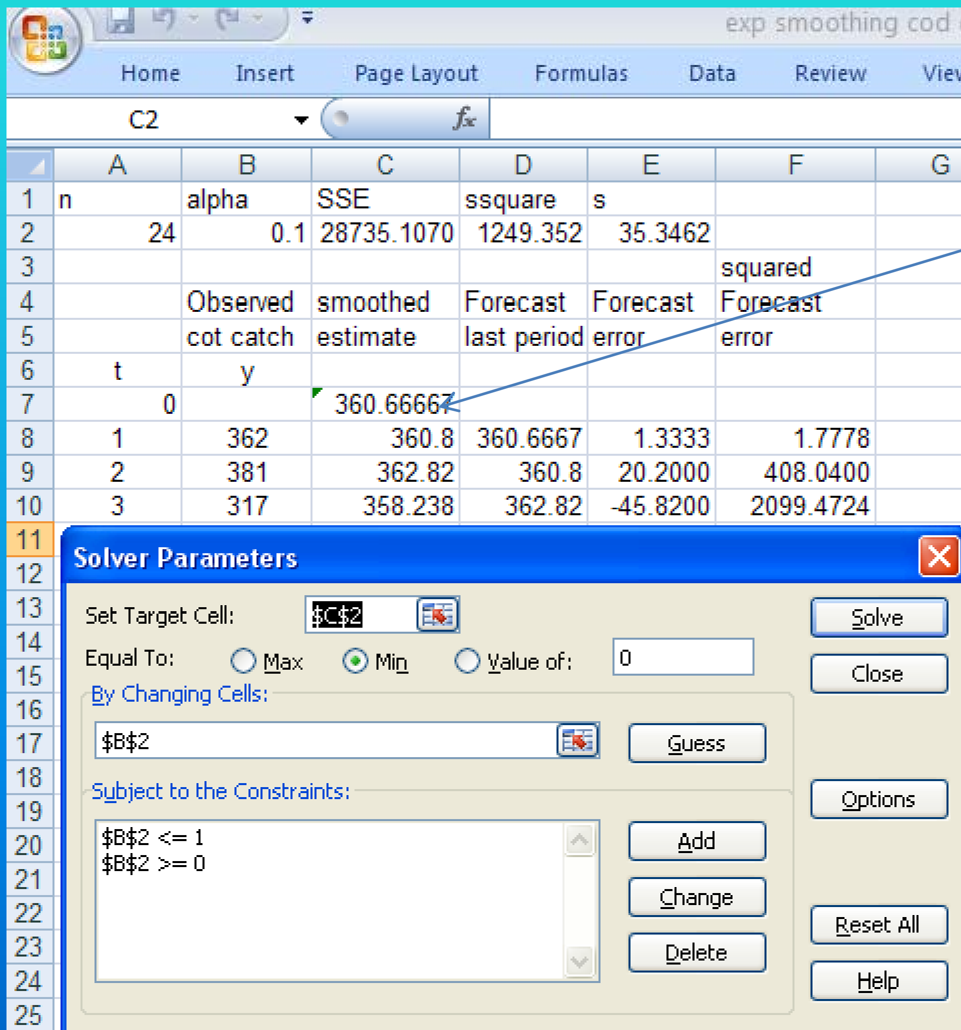
n	alpha	SSE	ssquare	s	
24	0.1	28735.1070	1249.352	35.3462	

t	Observed cod catch y	smoothed estimate	Forecast last period	Forecast error	squared Forecast error
0		360.66667			
1	362	360.8	360.6667	1.3333	1.7778
2	381	362.82	360.8	20.2000	408.0400
3	317	358.238	362.82	-45.8200	2099.4724
4	297	352.1142	358.238	-61.2380	3750.0926
5	399	356.80278	352.1142	46.8858	2198.2782
6	402	361.3225	356.8028	45.1972	2042.7887
7	375	362.69025	361.3225	13.6775	187.0740
8	349	361.32123	362.6903	-13.6903	187.4230
9	386	363.7891	361.3212	24.6788	609.0419
10	328	360.21019	363.7891	-35.7891	1280.8600
11	389	363.08917	360.2102	28.7898	828.8530
12	343	361.08026	363.0892	-20.0892	403.5749

t	Observed cod catch y	smoothed estimate	Forecast last period	Forecast error	squared Forecast error
13	276	352.57223	361.0803	-85.0803	7238.6501
14	334	350.71501	352.5722	-18.5722	344.9278
15	394	355.04351	350.715	43.2850	1873.5905
16	334	352.93916	355.0435	-21.0435	442.8292
17	384	356.04524	352.9392	31.0608	964.7760
18	314	351.84072	356.0452	-42.0452	1767.8023
19	344	351.05665	351.8407	-7.8407	61.4768
20	337	349.65098	351.0566	-14.0566	197.5893
21	345	349.18588	349.651	-4.6510	21.6316
22	362	350.46729	349.1859	12.8141	164.2016
23	314	346.82056	350.4673	-36.4673	1329.8636
24	365	348.63851	346.8206	18.1794	330.4919

Holt's Simple Exponential Smoothing, Example on page 348 – using Excel SOLVER to find optimum α so that SSE is minimum

In EXCEL: Data/solver, then enter the Solver Parameters as shown below – then click on SOLVE to obtain the results on the next slide.



	A	B	C	D	E	F	G
1	n	alpha	SSE	ssquare	s		
2	24	0.1	28735.1070	1249.352	35.3462		
3						squared	
4		Observed	smoothed	Forecast	Forecast	Forecast	
5		cot catch	estimate	last period	error	error	
6	t	y					
7	0		360.66667				
8	1	362	360.8	360.6667	1.3333	1.7778	
9	2	381	362.82	360.8	20.2000	408.0400	
10	3	317	358.238	362.82	-45.8200	2099.4724	

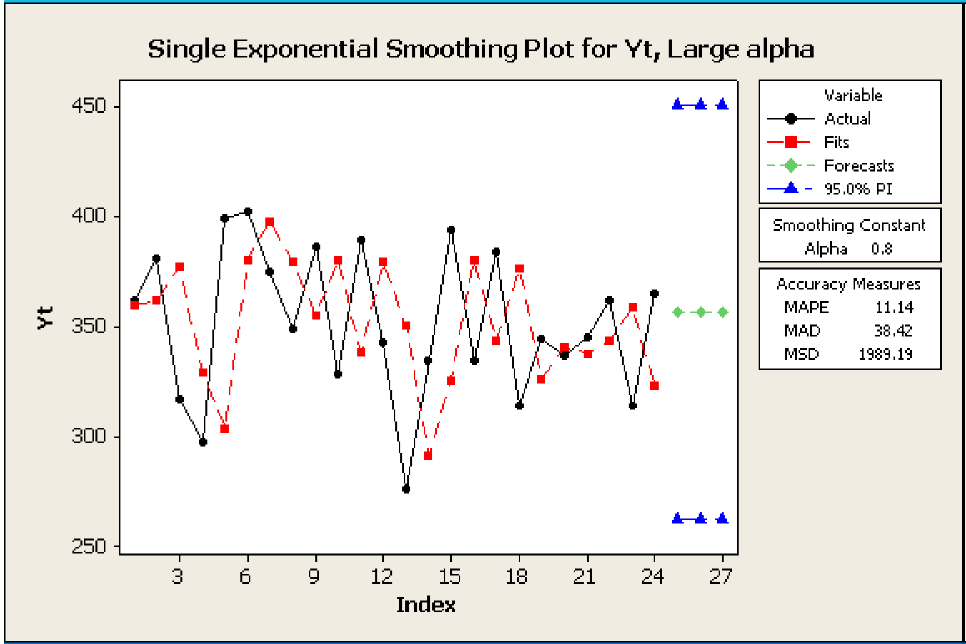
obtained by fitting least squares line yt = b0 + b1*t fitted to first 26 data points of the total 52 points

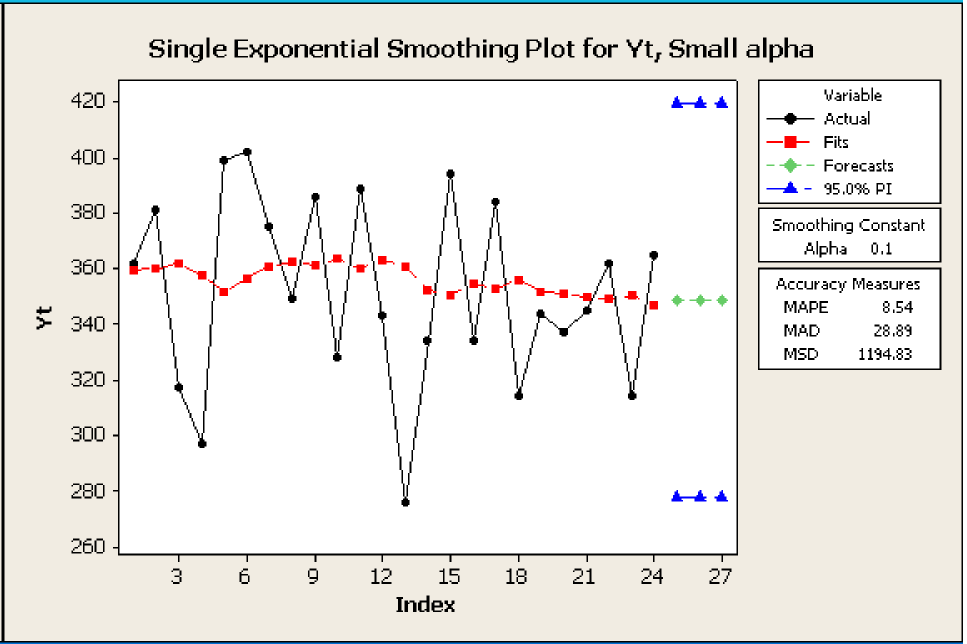
Holt's Exponential Smoothing, Optimal α for Example on page 348 – obtained from Excel SOLVER

	A	B	C	D	E	F
1	n	alpha	SSE	ssquare	s	
2	24	0.034353	28089.1409	1221.267	34.9466	
3						squared
4		Observed	smoothed	Forecast	Forecast	Forecast
5		cot catch	estimate	last period	error	error
6	t	y				
7	0		360.66667			
8	1	362	360.71247	360.6667	1.3333	1.7778
9	2	381	361.40941	360.7125	20.2875	411.5838
10	3	317	359.88381	361.4094	-44.4094	1972.1959
11	4	297	357.72355	359.8838	-62.8838	3954.3732
12	5	399	359.14153	357.7235	41.2765	1703.7456
13	6	402	360.61385	359.1415	42.8585	1836.8489
14	7	375	361.10806	360.6139	14.3861	206.9613
15	8	349	360.69211	361.1081	-12.1081	146.6052
16	9	386	361.56152	360.6921	25.3079	640.4893
17	10	328	360.40857	361.5615	-33.5615	1126.3755
18	11	389	361.39078	360.4086	28.5914	817.4698
19	12	343	360.759	361.3908	-18.3908	338.2208

REMARKS ON SIMPLE EXPONENTIAL SMOOTHING

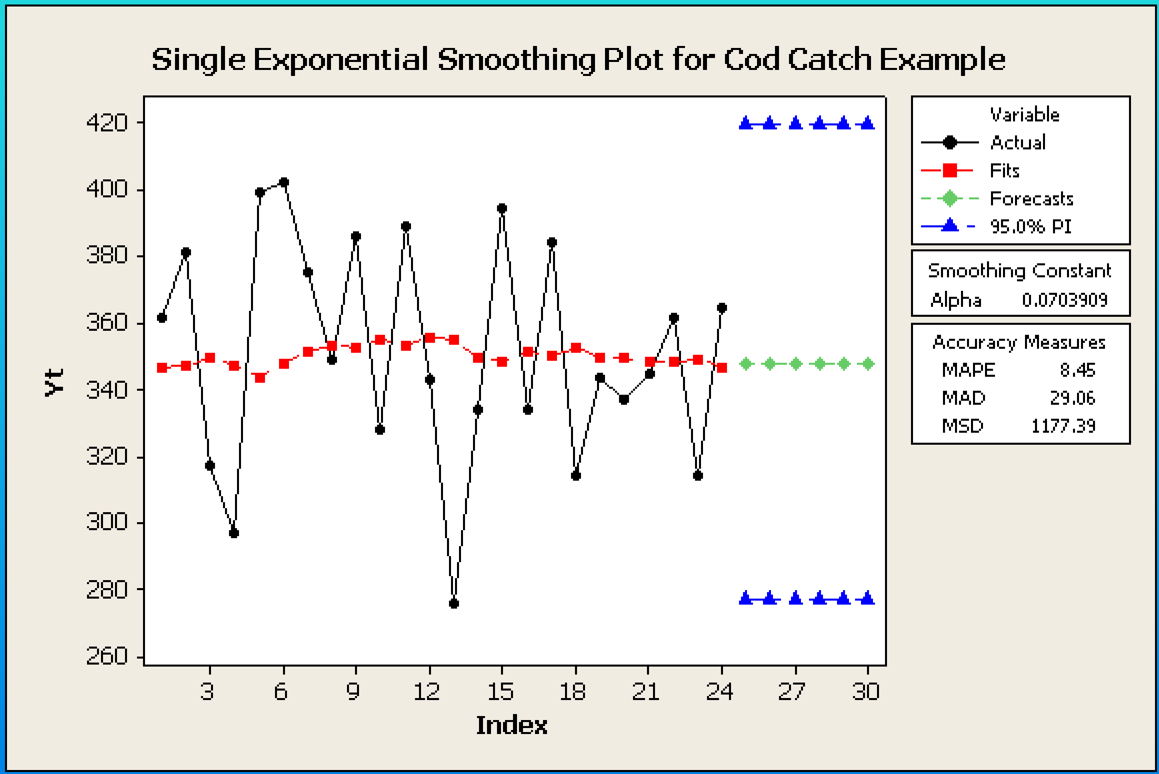
1) Large value of α gives more weight to last observation y_t and hence yields rapid changes in the fitted values. Small value of α gives a smoother look in the fitted values.





2) Minitab : Stat/Time Series/Single Exponential Smooting performs SES in a different way: it computes an optimal weight by fitting an ARIMA (0, 1, 1) model to the data. With this option, Minitab calculates the initial smoothed value by backcasting.

3) Minitab calculates the prediction intervals for forecasts using $\tau = 1$, and hence the prediction interval width does not change (see next slide), which is not correct. The width of the prediction interval should increase as τ is incresaed.


Single Exponential Smoothing Plot for Cod Catch Example
420
400
380
360
340
320
300
280
260
Yt
3
6
9
12
15
18
21
24
27
30
Index
Variable
Actual
Fits
Forecasts
95.0% PI
Smoothing Constant
Alpha 0.0703909
Accuracy Measures
MAPE 8.45
MAD 29.06
MSD 1177.39

Holt's Trend Corrected Exponential Smoothing or Double Exponential Smoothing

- If y_t exhibits a linear trend $y_t = \beta_0 + \beta_1 t$

i.e., level (mean) changes at a constant rate β_1 , then SES should be used.

- When both level and growth rate are changing, Holt's Trend Corrected Smoothing (DES) is appropriate.

NOTATIONS USED:

ℓ_{T-1} = estimate of level (mean) at time T -1

b_{T-1} = estimate of growth rate of y_t at time T -1

$$\ell_T = \alpha y_t + (1-\alpha)[\ell_{T-1} + b_{T-1}]$$

$$b_T = \gamma[\ell_T - \ell_{T-1}] + (1-\gamma)b_{T-1}$$

Adjustment for changing growth rate

where

α and γ are smoothing constants, $0 \le \alpha \le 1,\ 0 \le \gamma \le 1.$

Forecast made at time T for y_t is

$$\hat{y}_{T+\tau} = \ell_T + \tau b_t \ (T = 0,1,2,...,n-1; \tau = 1,2,3,...,n)$$

for historical data $(1 \le t \le n)$.

$$\ell_T = \alpha y_t + (1-\alpha)[\ell_{T-1} + b_{T-1}]$$

$$b_T = \gamma[\ell_T - \ell_{T-1}] + (1-\gamma)b_{T-1}$$

Forecast made at time T for y_t is

$$\hat{y}_{T+\tau} = \ell_T + \tau b_t \ (\tau = 1, 2, 3, \ldots, n)$$

$$T = 0:$$

$$\ell_1 = \alpha y_1 + (1-\alpha)[\ell_0 + b_0]$$

$$b_1 = \gamma[\ell_1 - \ell_0] + (1-\gamma)b_0$$

$$\hat{y}_1(0) = \ell_0 + 1b_0$$

growth rate b_0 and level ℓ_0 can be obtained by fitting a straight line to 1-st half of data,

b_0 = slope

ℓ_0 = intercept

Forecast made at time T=1 for y_2 :

$$\ell_2 = \alpha y_2 + (1-\alpha)[\ell_1 + b_1]$$

$$b_2 = \gamma[\ell_2 - \ell_1] + (1-\gamma)b_1$$

$$\hat{y}_2(1) = \ell_2 + b_2 \ (\tau=1)$$

Forecast made at time T=2 for y_3 :

$$\ell_3 = \alpha y_3 + (1-\alpha)[\ell_2 + b_2]$$

$$b_3 = \gamma[\ell_3 - \ell_2] + (1-\gamma)b_2$$

$$\hat{y}_3(2) = \ell_2 + b_2 \ (\tau=1)$$

…………

Prediction intervals for future observations:

$$\ell_2 = \alpha y_2 + (1-\alpha)[\ell_1 + b_1]$$

$$b_2 = \gamma[\ell_2 - \ell_1] + (1-\gamma)b_1$$

$$\hat{y}_{T+\tau}(T) = \ell_T + \tau b_T \ (\tau = 1,2,3,...; T = n)$$

If $\tau = 1$, a 95% prediction interval computed at time $T(=n)$ for $y_{T+1}(=y_{n+1})$ is

$$(\ell_T + b_T) \pm 1.96 \times s$$

If τ=2, a 95% prediction interval computed at time $T(=n)$ for $y_{T+2}(=y_{n+2})$ is

$$(\ell_T + 2b_T) \pm 1.96 \times s \times \sqrt{1+\alpha^2(1+\gamma)^2}$$

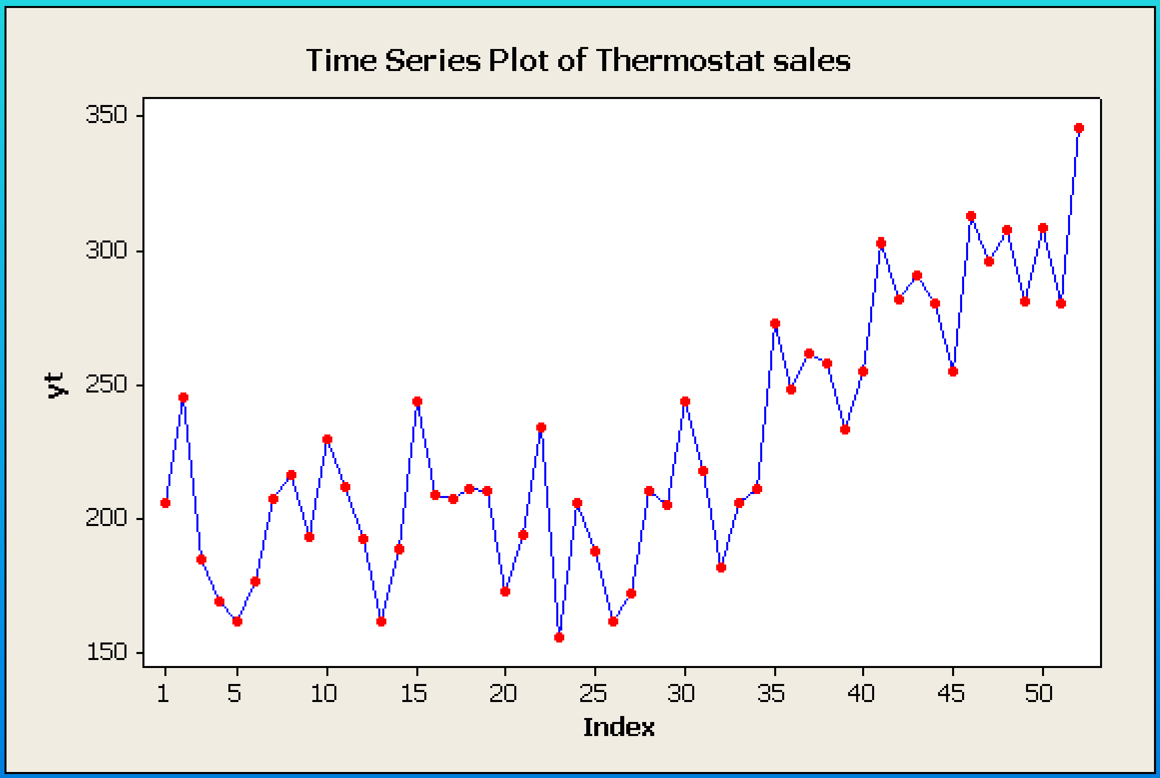
If τ=3, a 95% prediction interval computed at time $T(=n)$ for $y_{T+3}(=y_{n+3})$ is

$$(\ell_T + 3b_T) \pm 1.96 \times s \times \sqrt{1+\alpha^2(1+\gamma)^2 + \alpha^2(1+2\gamma)^2}$$

In general:

$$(\ell_T + \tau b_T) \pm 1.96 \times s \times \sqrt{1+\alpha^2(1+\gamma)^2 + \alpha^2(1+2\gamma)^2 + \ldots + \alpha^2[(1+2\gamma+\ldots+(\tau-1)\gamma^2]}$$

Example 8.3/page 359 of Bowerman, O'Connell, Koehler, 4-th Edition – Forecasting, Time Series, and Regression: weekly thermostat sales data



The regression equation fitted to first 26 data points is

yt = 203 - 0.368 t. To get betas to 4-th decimal, store coefficients:

COEF1
202.6246
-0.3682
To be used as initial estimates

Example 8.3/page 359 of , Bowerman, O'Connell, Koehler, 4-th Edition – Forecasting, Time Series, and Regression: weekly thermostat sales data

n	alpha	gamma	SSE	ssquare	s
52	0.2	0.1	39182.4700	783.6494	27.9937

t	observed thermostat sales y	level, LT	growth rate, BT	forecast made last period	forecast error	squared forecast error
0		202.6246	-0.3682			
1	206	203.0051	-0.293328	202.2564	3.7436	14.0145
2	245	211.1694	0.5524362	202.7118	42.2882	1788.2925
3	185	206.3775	0.0179988	211.7219	-26.7219	714.0583
4	169	198.9164	-0.7299111	206.3955	-37.3955	1398.4230
5	162	190.9492	-1.4536408	198.1865	-36.1865	1309.4617
6	177	186.9964	-1.7035518	189.4955	-12.4955	156.1387
7	207	189.6343	-1.2694095	185.2929	21.7071	471.1988
8	216	193.8919	-0.7167074	188.3649	27.6351	763.6988
9	193	193.1402	-0.7202117	193.1752	-0.1752	0.0307
10	230	199.936	0.0313892	192.42	37.5800	1412.2596
11	212	202.3739	0.2720421	199.9674	12.0326	144.7845
12	192	200.5167	-0.0591235	202.6459	-10.6459	113.3357

t	observed thermostat sales y	level, LT	growth rate, BT	forecast made last period	forecast error	squared forecast error
13	162	192.8607	-0.7123938	200.5759	-38.5759	1488.0973
14	189	191.5186	-0.7753597	192.1483	-3.1483	9.9118
15	244	201.3946	0.2897747	190.7433	53.2567	2836.2784
16	209	203.1475	0.4360868	201.6844	7.3156	53.5180
17	207	204.2669	0.5044147	203.5836	3.4164	11.6718
18	211	206.017	0.6289887	204.7713	6.2287	38.7967
19	210	207.3168	0.6960681	206.646	3.3540	11.2491
20	173	201.0103	-0.0041897	208.0129	-35.0129	1225.9025
21	194	199.6049	-0.1443121	201.0061	-7.0061	49.0858
22	234	206.3685	0.5464762	199.4606	34.5394	1192.9711
23	156	196.732	-0.4718227	206.9149	-50.9149	2592.3316
24	206	198.2081	-0.2770254	196.2601	9.7399	94.8650

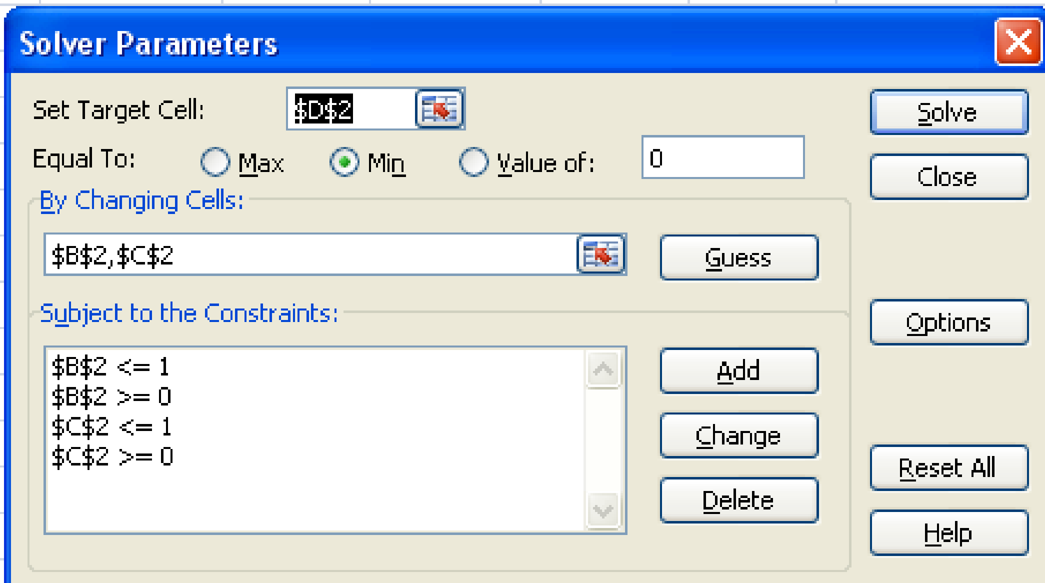
t	observed thermostat sales y	level, LT	growth rate, BT	forecast made last period	forecast error	squared forecast error
25	188	195.9449	-0.475647	197.9311	-9.9311	98.6264
26	162	188.7754	-1.1450314	195.4692	-33.4692	1120.1885
27	172	184.5043	-1.4576382	187.6303	-15.6303	244.3076
28	210	188.4373	-0.918571	183.0466	26.9534	726.4838
29	205	191.015	-0.5689457	187.5187	17.4813	305.5945
30	244	201.1568	0.5021334	190.446	53.5540	2868.0261
31	218	204.9272	0.828954	201.659	16.3410	267.0293
32	182	201.0049	0.3538314	205.7561	-23.7561	564.3537
33	206	202.287	0.4466567	201.3587	4.6413	21.5413
34	211	204.3869	0.6119838	202.7336	8.2664	68.3326
35	273	218.5991	1.9720058	204.9989	68.0011	4624.1497
36	248	226.0569	2.5205833	220.5711	27.4289	752.3432

t	observed thermostat sales y	level, LT	growth rate, BT	forecast made last period	forecast error	squared forecast error
37	262	235.262	3.1890337	228.5775	33.4225	1117.0646
38	258	242.3608	3.5800133	238.451	19.5490	382.1626
39	233	243.3527	3.3211967	245.9408	-12.9408	167.4651
40	255	248.3391	3.4877194	246.6739	8.3261	69.3246
41	303	262.0614	4.5111833	251.8268	51.1732	2618.6956
42	282	269.6581	4.8197307	266.5726	15.4274	238.0038
43	291	277.7823	5.150174	274.4778	16.5222	272.9820
44	280	282.346	5.0915252	282.9324	-2.9324	8.5992
45	255	280.95	4.4427756	287.4375	-32.4375	1052.1900
46	312	290.7142	4.9749204	285.3928	26.6072	707.9453
47	296	295.7513	4.9811379	295.6891	0.3109	0.0966
48	307	301.986	5.1064891	300.7324	6.2676	39.2823
49	281	301.874	4.5846403	307.0924	-26.0924	680.8155
50	308	306.7669	4.6154684	306.4586	1.5414	2.3759
51	280	305.1059	3.9878216	311.3823	-31.3823	984.8514
52	345	316.275	4.7059477	309.0937	35.9063	1289.2627

Holt's Trend Corrected Exponential Smoothing, Example on page 360 – using Excel SOLVER to find optimum α and γ so that SSE is minimum

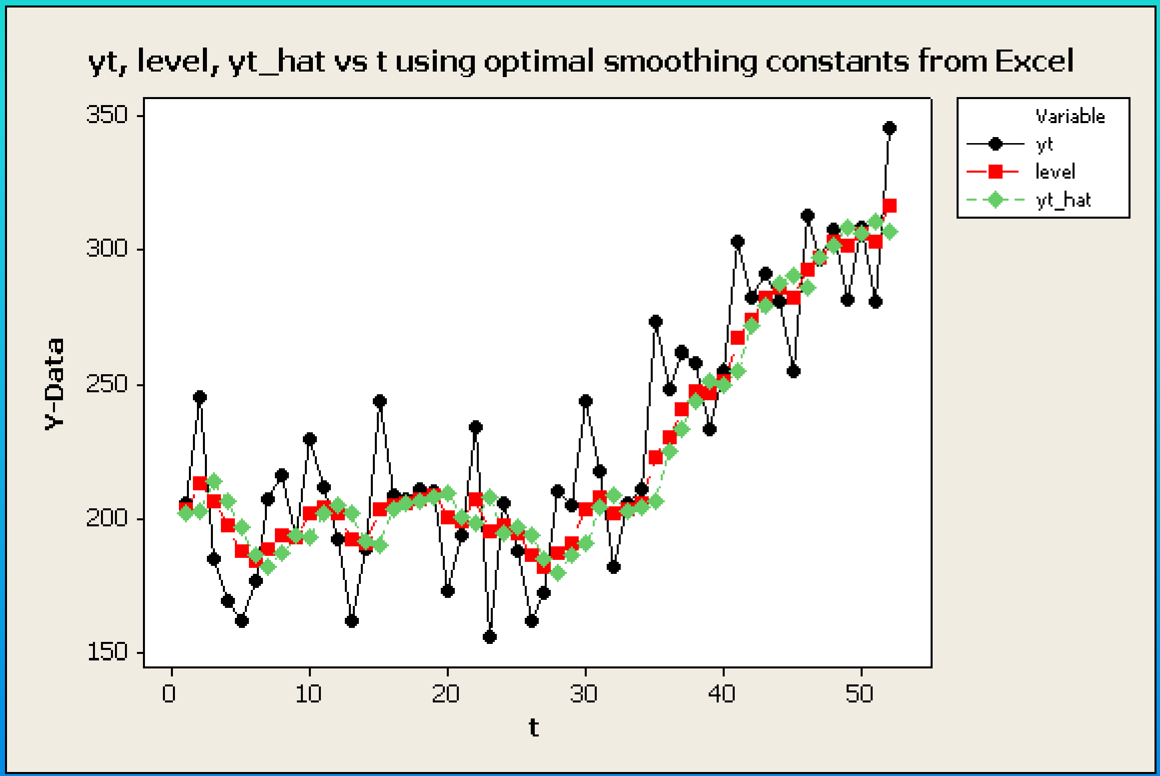
In EXCEL: Data/solver, then enter the Solver Parameters as shown below – click on SOLVE to obtain results shown on the next slide.

A	B	C	D	E	F	G
n	alpha	gamma	SSE		ssquare	s
52	0.2	0.1	39182.4700		783.6494	27.9937
	observed thermostat sales y	level, LT	growth rate, BT	forecast made last period	forecast error	squared forecast error
t						
0		202.6246	-0.3682			
1	206	203.0051	-0.293328	202.2564	3.7436	14.0145
2	245	211.1694	0.5524362	202.7118	42.2882	1788.2925
3						
4						
5						
6						
7						
8						
9						
10						
11						
12						
13						
14						
15						



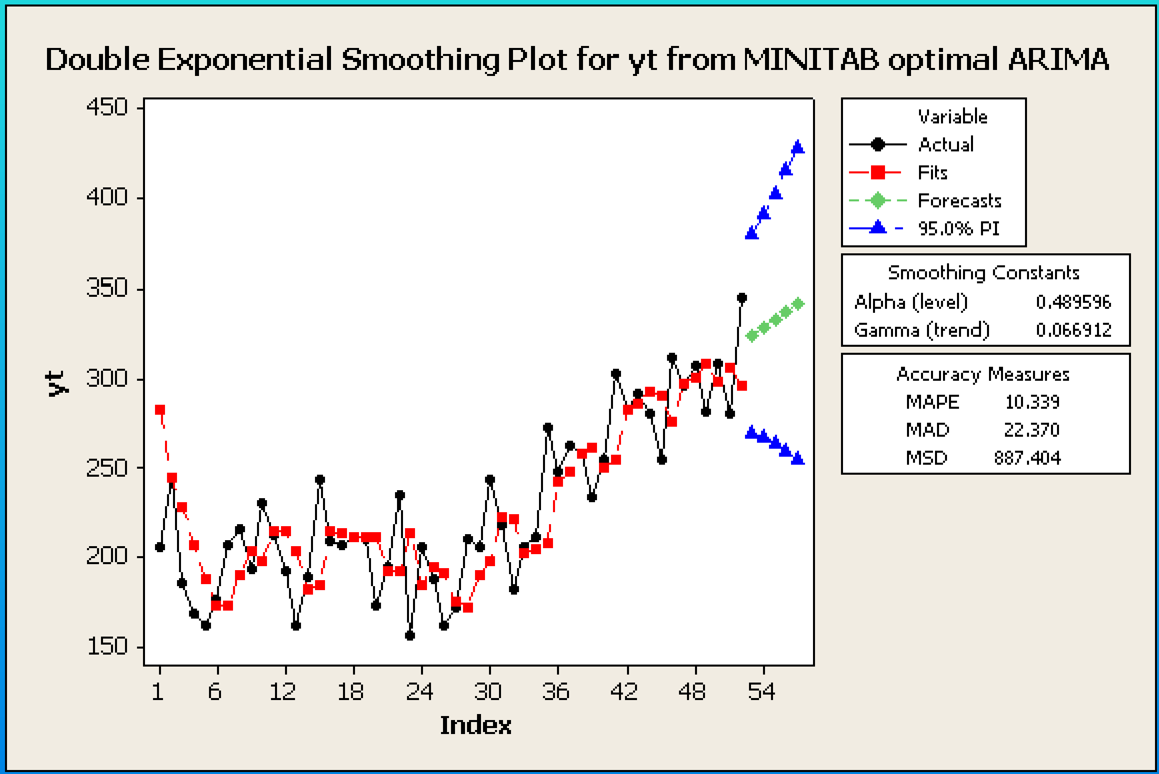
Holt's Trend Corrected Exponential Smoothing, Optimal α for Example on page 360 – obtained from Excel SOLVER

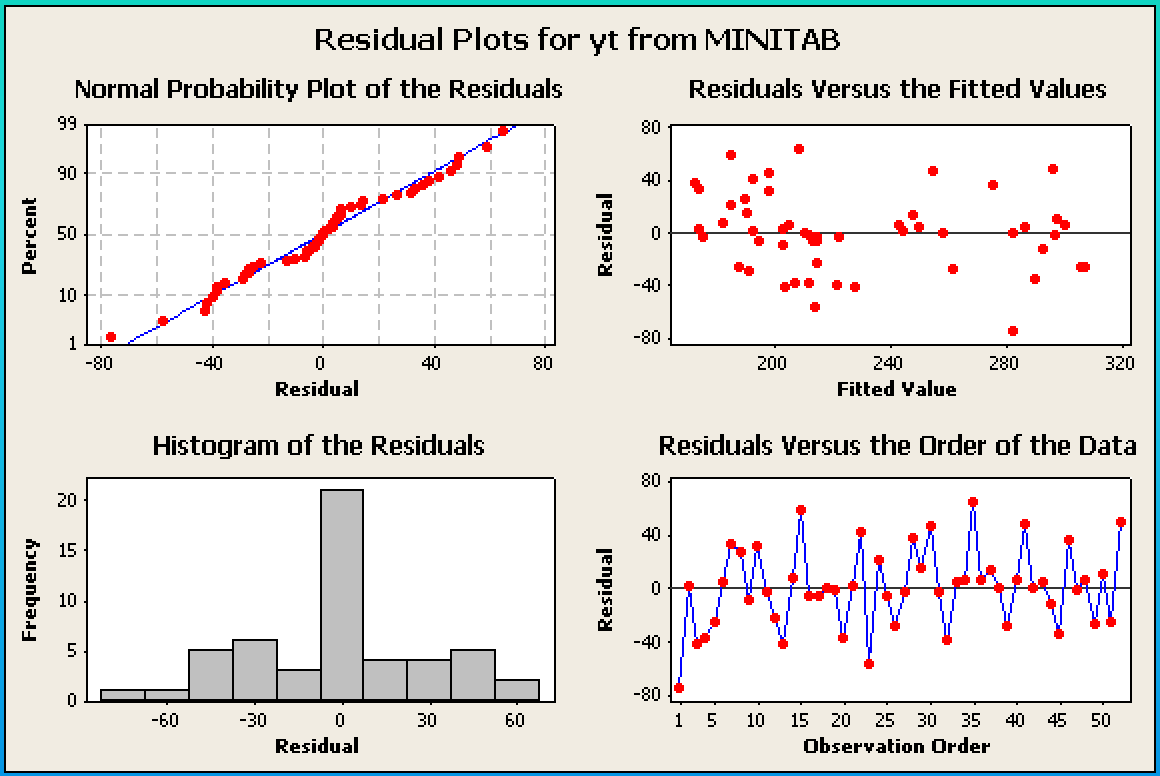
	A	B	C	D	E	F	G
1	n	alpha	gamma	SSE		ssquare	s
2	52	0.24684184	0.095055	38884.2444		777.6849	27.8870
3					forecast		
4		observed			made		squared
5		thermostat		growth	last	forecast	forecast
6	t	sales y	level, LT	rate, BT	period	error	error
7	0		202.6246	-0.3682			
8	1	206	203.1805	-0.2803622	202.2564	3.7436	14.0145
9	2	245	213.2921	0.707447	202.9001	42.0999	1772.4003
10	3	185	206.8413	0.0270165	213.9996	-28.9996	840.9753
11	4	169	197.5208	-0.8615046	206.8683	-37.8683	1434.0069
12	5	162	188.1039	-1.674732	196.6593	-34.6593	1201.2672
13	6	177	184.1017	-1.8959738	186.4292	-9.4292	88.9099



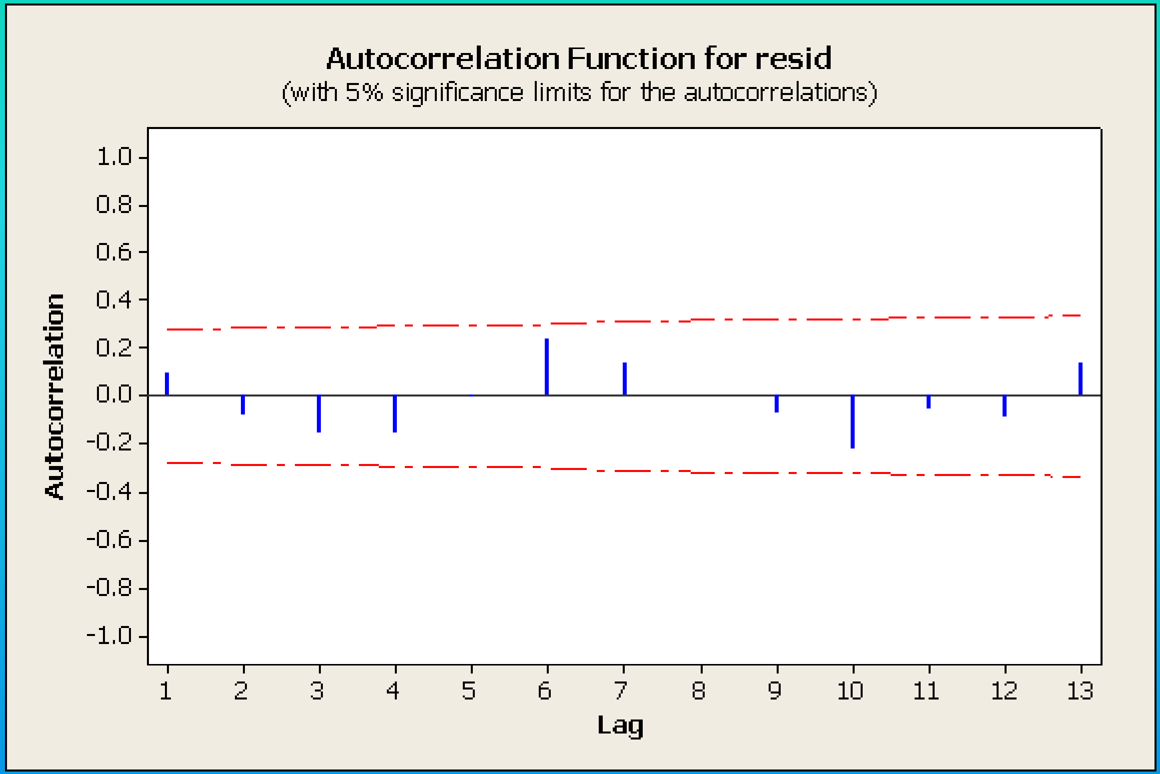
MSD = SSE/(n-2) = 777.68
MAD = 21.70

Notice that results calculated in Excel are slightly better than the results from MINITAB (next slide).


Double Exponential Smoothing Plot for yt from MINITAB optimal ARIMA
yt
Index
Variable
Actual
Fits
Forecasts
95.0% PI
Smoothing Constants
Alpha (level) 0.489596
Gamma (trend) 0.066912
Accuracy Measures
MAPE 10.339
MAD 22.370
MSD 887.404

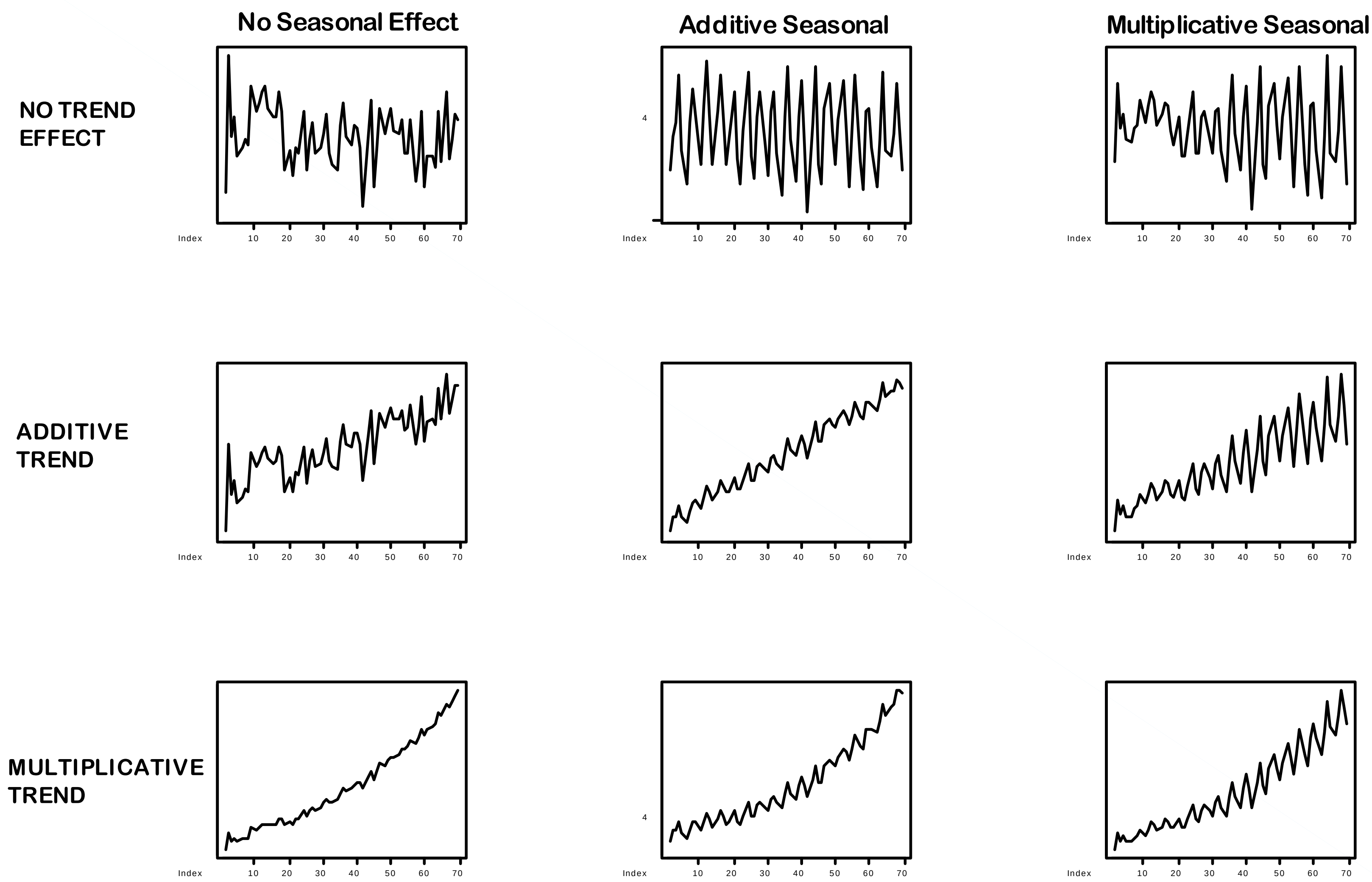


Residual plots indicate good fit.



Autocorrelation plot of residual shows residuals are not correlated..

Patterns based on Pegel's Classification

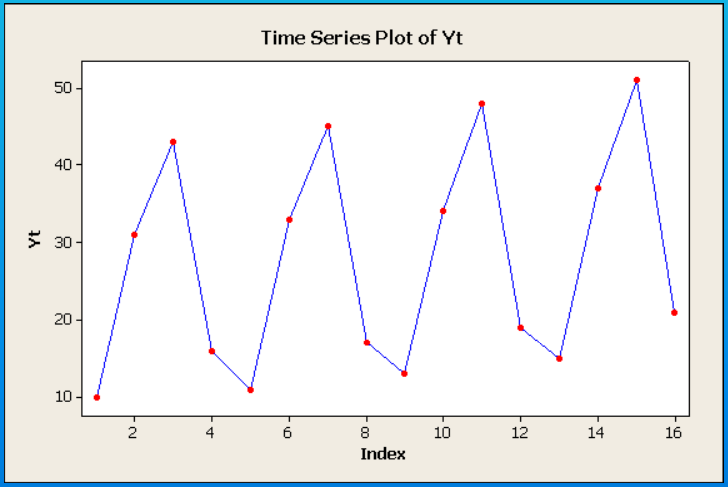


Trend Component ↓	**Seasonal Component** 1 (none)	2 (additive)	3 (multiplicative)
A (none)	A1 - SES		A3 HW-Mult
B (Additive)	B1 - DES(Holt's)	B2 - HW-Add	B3 HW-Mult
C (Multiplicative)	C1	C2	C3

HW = Holt-Winter's Method, SES = Simple Exponential Smoothing, DES = Double Exponential Smoothing (Holt's Trend Corrected Exponential Smoothing)

HOLT-WINTERS METHODS

Additive H-W method for time series with constant seasonal variation



Multiplicative H-W method for time series with increasing seasonal variation

If a time series has a linear trend with a constant growth rate β_1, fixed seasonal pattern SN_t with constant variation, then the time series can be described by the model

$$y_t = (\beta_0 + \beta_1 t) + SN_t + e_t$$

If the level (mean), growth rate, and the seasonal pattern of the time series are changing, then H-W method is appropriate.

At time T, let

l_T = estimated level

b_T = estimated growth

sn_T = estimated seasonal factor

L = number of seasons in a year

$L = 4$ for quarterly data,

$L = 12$ for monthlly data

H-W smoothing equations are:

$$l_T = \alpha(y_T - sn_{T-L}) + (1-\alpha)(l_{T-1} + b_T)$$

$$b_T = \gamma(l_T - l_{T-1}) + (1-\gamma)b_{T-1}$$

$$sn_T = \delta(y_T - l_T) + (1-\delta)sn_{T-L}$$

$$0 \le \alpha \le 1, 0 \le \gamma \le 1, 0 \le \delta \le 1$$

A point forecast in time period T

for $y_{T+\tau}$ is

$$\hat{y}_{T+\tau} = l_T) + \tau b_T + sn_{T+\tau-L}$$

most recent estimate of the seasonal factor for the season corresponding to time T+τ

NOTE: For historical data ($1 \leq T \leq n$)

$T = 0,1,2,3,...$ and $\tau=1$ as we predict for next time period using the previous values.

Once we have reached T = n, and are ready to forecast into future (when *T = n*) $\hat{y}_{T+\tau}(T) = \ell_T + b_T + sn_{T+\tau-L}, \tau = 1,,3,\ldots$

A 95% prediction interval computed at time T $(=n)$ is $\hat{y}_{T+\tau}(T) \pm 1.96 \times s \times \sqrt{c_\tau}$

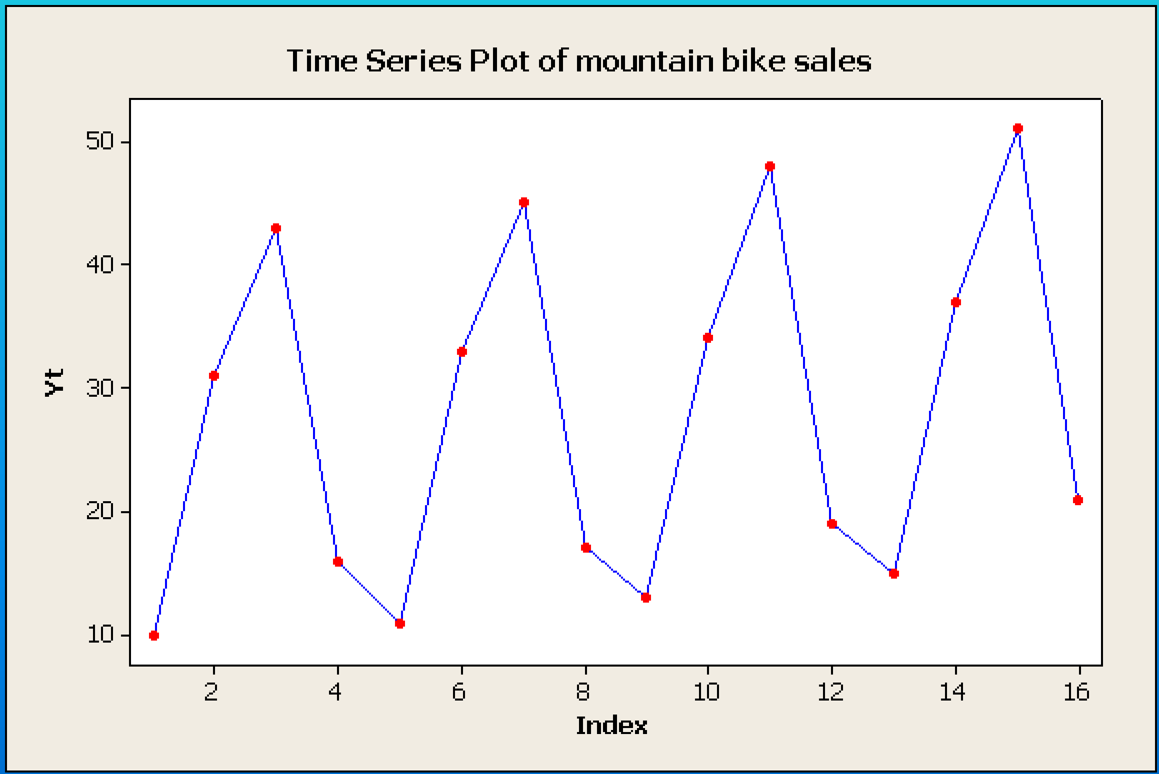
where

$$s = \sqrt{\frac{SSE}{T-3}}$$

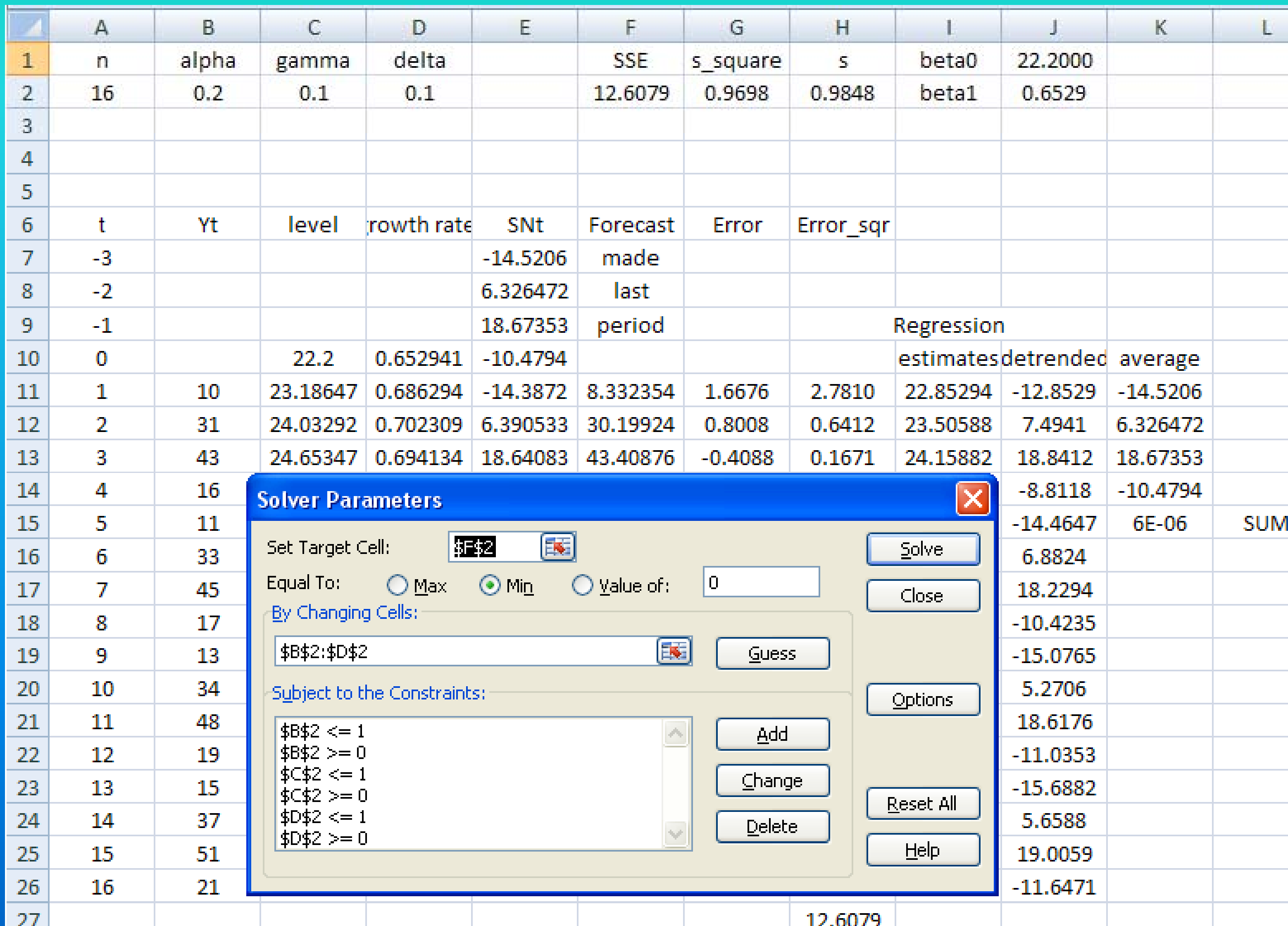
$$c_\tau = \begin{cases} 1, \quad \tau = 1 \\ [1 + \sum_{j=1}^{\tau-1} \alpha^2 (i + j\gamma)^2], \ 2 \le \tau \le L \\ 1 + \sum_{j=1}^{\tau-1} \alpha^2 (i + j\gamma)^2] + d_{j,L}(1-\alpha)\delta^2, \ \tau > L \end{cases}$$

$$\text{and } d_{j,L} = \begin{cases} 1 \text{ if } j \text{ is an integer multiple of } L \\ 0 \quad \text{otherwise} \end{cases}$$

Example 8.4 (Mountain Bike Sales data from Chapter 6, Table 6.8)
NOTE: Figure 8.9/page 370 has data errors in the last 8 lines (correct data is given in Table 6.8/page 318 of text)



Example 8.4 (Mountain Bike Sales data from Chapter 6, Table 6.8)
NOTE: Figure 8.9/page 370 has data errors in the last 8 lines (correct data is given in Table 6.8/page 318 of text)
In Excel, DATA/SOLVER, then add constraints to get following figure:



	A	B	C	D	E	F	G	H	I	J	K	L
1	n	alpha	gamma	delta		SSE	s_square	s	beta0	22.2000		
2	16	0.2	0.1	0.1		12.6079	0.9698	0.9848	beta1	0.6529		
3												
4												
5												
6	t	Yt	level	rowth rate	SNt	Forecast	Error	Error_sqr				
7	-3				-14.5206	made						
8	-2				6.326472	last						
9	-1				18.67353	period			Regression			
10	0		22.2	0.652941	-10.4794				estimates	detrended	average	
11	1	10	23.18647	0.686294	-14.3872	8.332354	1.6676	2.7810	22.85294	-12.8529	-14.5206	
12	2	31	24.03292	0.702309	6.390533	30.19924	0.8008	0.6412	23.50588	7.4941	6.326472	
13	3	43	24.65347	0.694134	18.64083	43.40876	-0.4088	0.1671	24.15882	18.8412	18.67353	
14	4	16								-8.8118	-10.4794	
15	5	11								-14.4647	6E-06	SUM
16	6	33								6.8824		
17	7	45								18.2294		
18	8	17								-10.4235		
19	9	13								-15.0765		
20	10	34								5.2706		
21	11	48								18.6176		
22	12	19								-11.0353		
23	13	15								-15.6882		
24	14	37								5.6588		
25	15	51								19.0059		
26	16	21								-11.6471		
27								12.6079				

Example 8.4 (Mountain Bike Sales data from Chapter 6, Table 6.8)
NOTE: Figure 8.9/page 370 has data errors in the last 8 lines (correct data is given in Table 6.8/page 318 of text) – OPTIMAL solution

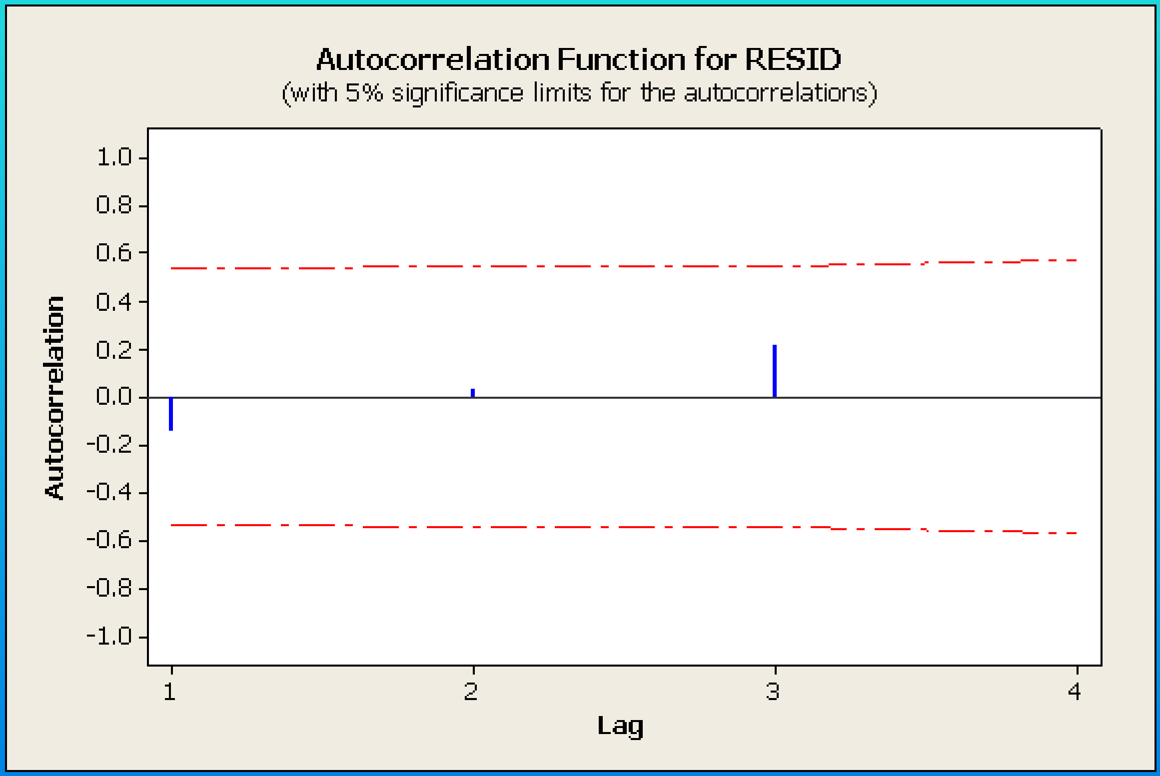
Holt-Winters example - Microsoft Excel

Home Insert Page Layout Formulas Data Review View Add-Ins

M1 f_x

	A	B	C	D	E	F	G	H	I	J	K	L
1	n	alpha	gamma	delta		SSE	s_square	s	beta0	22.2000		
2	16	0.369683	0	0		10.6857	0.8220	0.9066	beta1	0.6529		
3												
4												
5												
6	t	Yt	level	rowth rate	SNt	Forecast	Error	Error_sqr				
7	-3				-14.5206	made						
8	-2				6.326472	last						
9	-1				18.67353	period			Regression			
10	0		22.2	0.652941	-10.4794				estimates	detrendec	average	
11	1	10	23.46944	0.652941	-14.5206	8.332354	1.6676	2.7810	22.85294	-12.8529	-14.5206	
12	2	31	24.32613	0.652941	6.326472	30.44885	0.5511	0.3038	23.50588	7.4941	6.326472	
13	3	43	24.73782	0.652941	18.67353	43.6526	-0.6526	0.4259	24.15882	18.8412	18.67353	
14	4	16	25.79321	0.652941	-10.4794	14.91135	1.0887	1.1852	24.81176	-8.8118	-10.4794	
15	5	11	26.10399	0.652941	-14.5206	11.92557	-0.9256	0.8567	25.46471	-14.4647	6E-06	SUM
16	6	33	26.7261	0.652941	6.326472	33.0834	-0.0834	0.0070	26.11765	6.8824		
17	7	45	26.98992	0.652941	18.67353	46.05257	-1.0526	1.1079	26.77059	18.2294		
18	8	17	27.58244	0.652941	-10.4794	17.16345	-0.1635	0.0267	27.42353	-10.4235		
19	9	13	27.97113	0.652941	-14.5206	13.71479	-0.7148	0.5109	28.07647	-15.0765		
20	10	34	28.27267	0.652941	6.326472	34.95054	-0.9505	0.9035	28.72941	5.2706		
21	11	48	29.0738	0.652941	18.67353	47.59914	0.4009	0.1607	29.38235	18.6176		
22	12	19	29.63531	0.652941	-10.4794	19.24733	-0.2473	0.0612	30.03529	-11.0353		
23	13	15	30.00446	0.652941	-14.5206	15.76766	-0.7677	0.5893	30.68823	-15.6882		
24	14	37	30.66336	0.652941	6.326472	36.98387	0.0161	0.0003	31.34117	5.6588		

Autocorrelation function for RESID = Yt – Yt_hat from Holt-Winters Method



Prediction Intervals for t = 17, 18, 19 using T = 16

tau	c_tau
1	1
2	1.136666
3	1.136666

t	Yt	level	growth rate	SNt	Forecast		
16	21	32.02355	0.652941	-10.4794	21.86328	PL95	PU95
17					18.1559	16.37891	19.93289
18					39.6559	37.76137	41.43289
19					52.6559	50.76137	54.43289

MULTIPLICATIVE HOLT-WINTERS METHOD

The multiplicative Holt-Winters method is appropriate for a times series which has:

Linear trend, and a multiplicative seasonal factor *which are not fixed but change with time*

$$y_t = TR_t \times SN_t \times IR_t = [\beta_0 + t\beta_1(t)] \times SN_t \times IR_t$$

level ℓ_t

MULTIPLICATIVE HOLT-WINTERS METHOD:

$$\ell_T = \alpha \times (\text{deseasonalized } y_t) + (1-\alpha) \times (\text{adjusted level})$$

$$\ell_T = \alpha \times (y_T / sn_{T-L}) + (1-\alpha) \times (\ell_{T-1} + b_{T-1})$$

$$b_T = \gamma(\ell_T - \ell_{T-1}) + (1-\gamma) b_{T-1}$$

$$sn_T = \delta \times (\text{current estimate of } SN_T) + (1\text{-}\delta) \times (\text{ previous estimate of } SN_T)$$

$$sn_T = \delta(y_T / \ell_T) + (1\text{-}\delta)\ sn_{T-L}$$

An approximate 95% prediction interval computed

at time T for $y_{T+\tau}$ is

$$\hat{y}_{T+\tau}(T) \pm 1.96 \times s_r \times \sqrt{c_\tau} \times sn_{T+\tau-L} \text{ where}$$

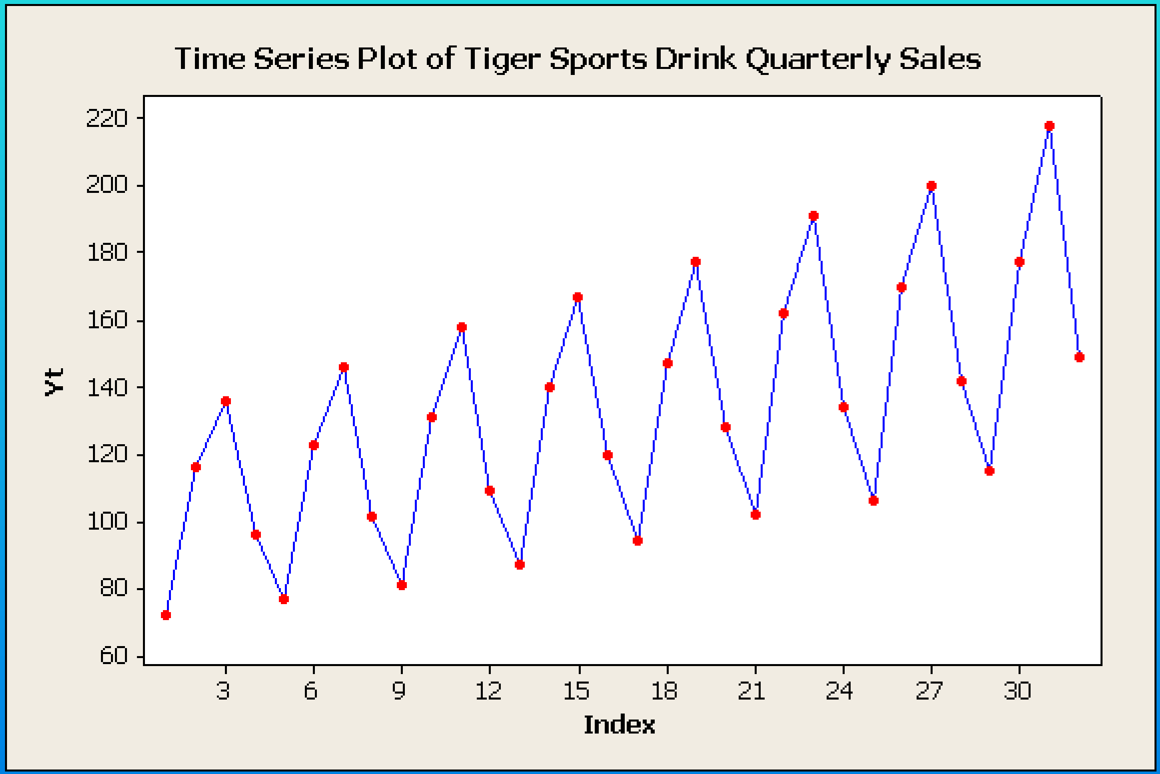
$$s_r = \text{relative standard error} = \sqrt{\frac{\sum_{t=1}^{T} \left[\frac{y_t - \hat{y}_t(t-1)}{\hat{y}_t(t-1)} \right]^2}{T-3}}$$

$$c_\tau = (\ell_T + b_T)^2, \ \tau = 1$$

$$c_\tau = \alpha^2(1+\gamma)^2(\ell_T + b_T)^2 + (\ell_T + 2b_T)^2, \ \tau = 2$$

$$c_\tau = \alpha^2(1+2\gamma)^2(\ell_T + b_T)^2 + \alpha^2(1+\gamma)^2(\ell_T + 2b_T)^2 + (\ell_T + 3b_T)^2, \ \tau = 3$$

Example 8.5/page 377 of text: Yt = quarterly sales (1000 cases) of Tiger Sports Drink



Since seasonal factor appears to be increasing, multiplicative Holt-Winters method is appropriate.

From regression line from 1-st 8 points

From regression estimates (see next slide)

n	alpha	gamma	delta	SSE	ssquare	s	SSRE	sr
32	0.2	0.1	0.1	177.3154	6.114324	2.472716	0.004275	0.065384

t	Yt	Level	Growth Rate	SNt	Forecast made last period	Forecast error	Squared Forecast error	Squared Relative error
-3				0.706224				
-2				1.111388				
-1				1.293693				
0		95.25	2.4706	0.888586				
1	72	98.56661	2.555201	0.70864	69.01264	2.987355	8.924292	0.016722
2	116	101.7722	2.620245	1.114229	112.3855	3.614461	13.06433	0.013513
3	136	104.5391	2.634903	1.294419	135.0518	0.948154	0.898996	4.43E-05
4	96	107.3466	2.652161	0.889157	95.23325	0.766748	0.587902	3.81E-05
5	77	109.7305	2.625336	0.70795	77.95045	-0.95045	0.903362	0.000134
6	123	111.9627	2.586025	1.112664	125.1901	-2.1901	4.796522	0.001468

t	Yt	Regression estimates	Detrended	SN_bar
1	72	97.7206	0.736794	0.706224
2	116	100.1912	1.157786	1.111388
3	136	102.6618	1.324738	1.293693
4	96	105.1324	0.913134	0.888586
5	77	107.603	0.715593	
6	123	110.0736	1.117434	
7	146	112.5442	1.297268	
8	101	115.0148	0.878148	
9	81	117.4854	0.689447	
10	131	119.956	1.092067	
11	158	122.4266	1.290569	
12	109	124.8972	0.872718	
13	87	127.3678	0.683061	
14	140	129.8384	1.078263	
15	167	132.309	1.262197	
16	120	134.7796	0.890342	

t	Yt	Level	Growth Rate	SNt	Forecast made last period	Forecast error	Squared Forecast error	Squared Relative error
7	146	114.1974	2.550889	1.292826	148.274	-2.274	5.17107	0.001216
8	101	116.1168	2.487739	0.887223	103.8075	-2.80753	7.882212	0.005766
9	81	117.7664	2.403928	0.70594	83.96674	-2.96674	8.801541	0.010988
10	131	119.6833	2.35523	1.110853	133.7092	-2.70918	7.339664	0.003013
11	158	122.0734	2.358718	1.292973	157.7746	0.22542	0.050814	1.04E-07
12	109	124.1168	2.32718	0.886321	110.399	-1.39903	1.957293	0.000314
13	87	125.8031	2.263098	0.704502	89.26191	-2.26191	5.116235	0.003285
14	140	127.6588	2.222359	1.109435	142.2628	-2.26279	5.120203	0.001295
15	167	129.7369	2.207928	1.292398	167.9329	-0.93293	0.870354	2.69E-05
16	120	132.6341	2.276854	0.888163	116.9455	3.05454	9.330216	0.006365
17	94	134.6143	2.247187	0.703881	95.04504	-1.04504	1.092104	0.000132
18	147	135.9891	2.159955	1.106589	151.8389	-4.8389	23.41499	0.023781
19	177	137.9102	2.136067	1.291503	178.5436	-1.54361	2.382735	0.000178

t	Yt	Level	Growth Rate	SNt	Forecast made last period	Forecast error	Squared Forecast error	Squared Relative error
20	128	140.8605	2.217494	0.890217	124.384	3.616029	13.07567	0.011051
21	102	143.4446	2.254151	0.704601	100.7099	1.290078	1.664301	0.000273
22	162	145.8382	2.268093	1.107012	161.2286	0.77143	0.595105	1.36E-05
23	191	148.063	2.263763	1.291352	191.2796	-0.27964	0.0782	1.67E-07
24	134	150.3664	2.26773	0.890311	133.8234	0.176584	0.031182	5.43E-08
25	106	152.1953	2.223844	0.703788	107.5461	-1.54609	2.390395	0.000494
26	170	154.2486	2.206793	1.106522	170.9438	-0.94377	0.890706	2.71E-05
27	200	156.1396	2.175215	1.290307	202.0389	-2.03893	4.157238	0.000423
28	142	158.5508	2.198815	0.890841	140.9494	1.050561	1.103678	6.13E-05
29	115	161.28	2.251852	0.704714	113.1337	1.866349	3.483259	0.000948
30	177	162.8176	2.180428	1.104581	180.9517	-3.95165	15.61558	0.007447
31	218	165.7889	2.259508	1.292769	212.8981	5.101872	26.0291	0.014948
32	149	167.8902	2.243694	0.890505	149.7044	-0.70438	0.496156	1.1E-05

Optimal α, γ, δ obtained by minimizing SSE (using Excel SOLVER) under the constraints $0 < \alpha < 1$, $0 < \gamma < 1$, $0 < \delta < 1$

n	alpha	gamma	delta	SSE	ssquare	s	SSRE	sr
32	0.33565	0.04548	0.133923	168.472	5.809378	2.410265	0.004063	0.06374

t	Yt	Level	Growth Rate	SNt	Forecast made last period	Forecast error	Squared Forecast error	Squared Relative error
-3				0.706224				
-2				1.111388				
-1				1.293693				
0		95.25	2.4706	0.888586				
1	72	99.14041	2.535173	0.708905	69.01264	2.987355	8.924292	0.016722
2	116	102.5813	2.576366	1.113989	113.001	2.999001	8.99401	0.006335
3	136	105.1468	2.575873	1.293658	136.0418	-0.04176	0.001744	1.64E-10
4	96	107.8282	2.580669	0.888816	95.72085	0.279147	0.077923	6.63E-07
5	77	109.8078	2.553334	0.707877	78.26938	-1.26938	1.611324	0.000424

Prediction Intervals from Holt-Winters Method

32	149	168.1218	2.302982	0.890798	150.3254	PL95	PU95	tau	c_tau	sqrt(c_tau)
33					120.0513	105.0533	135.0493	1	29044.62	170.4248
34					188.1113	162.9057	213.3169	2	33411.49	182.7881
35					220.4186	188.8367	252.0005	3	38204.23	195.459